Flipper games for monadically stable graph classes

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A pinnacle of sparsity theory, initiated by Nešetřil and Ossona de Mendez [NdM12], is the result of Grohe, Kreutzer, and Siebertz which identifies subgraf-closed classes of graphs with FPT first-order model checking as exactly *nowhere dense* classes [GKS17], which encompass most of the studied concepts of sparsity in graphs, including classes of graphs that exclude a fixed (topological) minor. One of the key steps in the proof is characterizing nowhere dense classes of graphs in terms of a combinatorial game called the *Splitter game*.

There is an ongoing effort aimed at generalizing sparsity theory to classes of graphs that are not necessarily sparse, e.g. *monadically stable* classes of graphs. Monadic stability is a notion of logical tameness for classes of structures. Introduced by Baldwin and Shelah [BS85] in the context of model theory, it has recently attracted attention in the field of structural graph theory. A class of graphs $\mathcal{C}$ is monadically stable if for any unary expansion $\widehat{\mathcal{C}}$ of $\mathcal{C}$ (i.e. adding arbitrary unary predicates on vertices of graphs in $\mathcal{C}$), one cannot interpret, in first-order logic, arbitrarily long linear orders in graphs from $\widehat{\mathcal{C}}$. It is known that nowhere dense graph classes are monadically stable [AA14]. On the other hand, monadic stability is a property expressed in purely model-theoretic terms and hence it is also suited for capturing structure in dense graphs.

It has been suspected that one can construct a structure theory for monadically stable graph classes that mirrors the theory of nowhere dense graph classes in the dense setting. Recently, a next step in this direction was provided by giving a characterization of monadic stability through the *Flipper game* [GMM+23]. This is a game on a graph played by *Flipper*, who in each round can complement the edge relation between any pair of vertex subsets, and *Connector*, who in each round is forced to localize the game to a ball of bounded radius. This is an analog of the *Splitter game*.

In [GMM+23] two different proofs of this result are given. The first proof is based on tools borrowed from model theory, and it exposes an additional property of monadically stable graph classes that is close in spirit to definability of types. The second proof relies on the recently introduced notion of *flip-wideness* [DMST22] and provides an efficient algorithm to compute Flipper's moves in a winning strategy.

The characterization of monadically stable classes of graphs in terms of the Flipper game was used to prove that model-checking is FPT on structurally nowhere dense classes of graphs [DMS23].

In the talk I will give an introduction to monadically stable classes of graphs and the Flipper game. I will also sketch the ideas behind the model-theoretic proof of the characterization. The talk will be mainly based on [GMM+23].

References

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