Second-Order Quantifiers with Poly-Logarithmic Bounds (extended abstract)

Shiguang FENG, Kexu WANG, and Xishun ZHAO
Sun Yat-sen University

Descriptive complexity theory is the study of the logical characterizations of computational complexity classes, which offers us a unique perspective on complexity theory. While finding the logic that captures P, or PTIME logic, remains a primary objective in this theory, our research focuses on the limited nondeterminism classes, especially the class βP. The *limited nondeterminism classes* refer to a specific category of computational complexity classes where the amount of non-determinism is restricted or constrained in some way. For the class βP, it consists of the problems whose computation's amount of nondeterminism is limited by a poly-logarithmic function w.r.t. the length of the input while whose running time is polynomial [8]. βP is often discussed for it lies between P and NP. As Fagin's theorem has linked the second-order quantification with the nondeterminism in computation [4], it is natural for us to consider adding some poly-logarithmic functions as restrictions on the second-order quantifiers. Let's call them *log-quantifiers*. The syntax and semantics of log-quantifiers first occurred in the publication of Ferrarotti et al [5]. While nearly at the same time, we independently introduced the log-quantifiers in our own paper [10]. Thus in our study, we accomplish three key objectives:

1. Using log-quantifiers, we define a series of new logics, $\mathrm{SO}^{\mathrm{plog}}$-$\mathcal{L}$, which we also call log-quantifier logics.

2. We show that the existential fragments capture the corresponding Guess-then-Check complexity classes on ordered structures, where especially, Σ_1^{plog}-IFP captures βP.

3. We study the expressive power of log-quantifiers, with the help of the classical methods: game method and 0-1 laws.

1 The log-quantifiers

Let $\mathcal{L}$ be a logic. Denote by $\mathrm{SO}^{\mathrm{plog}}$-$\mathcal{L}$ the minimal set that consists of

1. all the formulas of $\mathcal{L}$;

2. the formulas in the form $\exists^{\log^k} X\psi$ or $\forall^{\log^k} X\psi$, for any $k \in \mathbb{N}_+$ and relation variable X, if ψ is in the set.

Let Σ_1^{plog}-$\mathcal{L}$ (resp. Π_1^{plog}-$\mathcal{L}$) be the set of formulas in the form $\exists^{\log^{k_1}} X_1 \dots \exists^{\log^{k_m}} X_m\psi$ (resp. $\forall^{\log^{k_1}} X_1 \dots \forall^{\log^{k_m}} X_m\psi$), where ψ is a $\mathcal{L}$-formula. We recursively define $\Sigma_{n+1}^{\mathrm{plog}}$-$\mathcal{L} = \Sigma_1^{\mathrm{plog}}$-$(\Pi_n^{\mathrm{plog}}$-$\mathcal{L})$ and $\Pi_{n+1}^{\mathrm{plog}}$-$\mathcal{L} = \Pi_1^{\mathrm{plog}}$-$(\Sigma_n^{\mathrm{plog}}$-$\mathcal{L})$. Let $\mathcal{A}$ be an arbitrary structure. The satisfaction relation of $\mathrm{SO}^{\mathrm{plog}}$-$\mathcal{L}$ inherits that of $\mathcal{L}$, and

$$\mathcal{A} \models \exists^{\log^k} Y\psi \iff \exists R \subseteq A^{\mathrm{ar}(Y)} \text{ such that } |R| \leq \log^k(|\mathcal{A}|) \text{ and } \mathcal{A} \models \psi[R].$$

There are important four parameters. Let $sqr(\phi)$ (resp. $fqr(\phi)$) be the second-order (resp. first-order) quantifier rank of ϕ. Let $\text{mva}(\phi) = \max\{\text{ar}(X) \mid X \text{ is a bound variable } \phi\}$, which is the *maximal variable arity* of ϕ. And let $\text{height}(\phi) = \max\{k \mid \exists^{\log^k} \text{ or } \forall^{\log^k} \text{ occurs in } \phi\}$, which we call the *height* of ϕ. For any n, k, let $\Sigma_n^{\log^k}\text{-}\mathcal{L} = \{\phi \in \Sigma_n^{\text{plog}}\text{-}\mathcal{L} \mid \text{height}(\phi) \leq k\}$ and "$\Pi_n^{\log^k}\text{-}\mathcal{L}$" and "$\text{SO}^{\log^k}\text{-}\mathcal{L}$" are defined analogously.

Proposition 1.1. For any logic $\mathcal{L} \geq \text{FO}$ and natural numbers $n, k \geq 1$. Every formula ϕ of $\Sigma_n^{\log^k}\text{-}\mathcal{L}$ (resp. $\Pi_n^{\log^k}\text{-}\mathcal{L}$) is equivalent to a formula ϕ' of $\Sigma_n^{\log^k}\text{-}\mathcal{L}$ (resp. $\Pi_n^{\log^k}\text{-}\mathcal{L}$) such that $\text{mva}(\phi') \leq 2$.

Note that for the classical second-order logic, it is an open problem whether every Σ_1^1-formula is equivalent to a Σ_1^1-formula all of whose relation variables are at most binary [6].

2 Logical characterizations of the Guess-then-Check classes

Definition 2.1. [1] Let $g : \mathbb{N} \mapsto \mathbb{N}$ be a function and $\mathcal{C}$ a complexity class. A language L is in the class $GC(g, \mathcal{C})$ if there is a language $L' \in \mathcal{C}$ with an integer $c > 0$ such that for any $\mathcal{X}$,

$$\mathcal{X} \in L \Longleftrightarrow \exists \mathcal{Y} \in \{0,1\}^{\leq c\cdot g(|\mathcal{X}|)} \text{ and } \mathcal{X}\#\mathcal{Y} \in L',$$

where $\{0,1\}^{\leq c\cdot g(|\mathcal{X}|)}$ is the set of 01 strings of length at most $c\cdot g(|\mathcal{X}|)$.

By the classical results in descriptive complexity theory[7], it is easy to see that

Theorem 2.2. Let $\mathcal{L} \in \{\text{FOB}, \text{DTC}, \text{TC}, \text{IFP}\}$. On ordered structures, if $\mathcal{L}$ captures a complexity class $\mathcal{C}$, then $\Sigma_1^{\log^k}\text{-}\mathcal{L}$ captures $GC(\log^{k+1}, \mathcal{C})$.

Because for $k \in \mathbb{N}$, define $\beta_k = GC(\log^k, \text{P})$ and $\beta\text{P} = \bigcup_{k\geq 1} \beta_k$ [2], we have

Corollary 2.3. On ordered structures, $\Sigma_1^{\log^k}$-IFP captures β_{k+1} and Σ_1^{plog}-IFP captures βP.

Similar to the case of IFP, we have proved earlier in [10] that EVEN is not definable in SO^{plog}-IFP. Hence, EVEN is not definable in any logic weaker than SO^{plog}-IFP.

3 The expressive power of log-quantifiers

A logic satisfies the 0-1 law if for every formula of the logic, its asymptotic probability is either 0 or 1. FO satisfies the 0-1 law while MSO does not [3]. By a similar argument, we have the following proposition.

Proposition 3.1. There is a $\Sigma_1^{\log^2}$-FO formula that defines an order on almost all finite structures containing a binary relation.

By the fact that DTC defines EVEN on ordered structures, we have

Corollary 3.2. $\Sigma_1^{\log^2}$-DTC and $\Pi_1^{\log^2}$-DTC do not satisfy the 0-1 laws.

It is well known that the languages definable in the monadic second-order logic, MSO, are exactly the regular languages. As these logics share the idea of restricting the second-order variable, what are their differences? Let $\mathbb{A}^+$ is the set all the nonempty strings over a vocabulary $\mathbb{A}$ and $\mathbb{A}^* = \mathbb{A}^+ \cup \{\epsilon\}$. Define an equivalence relation $\equiv_{m,r,k,l}$ on $\mathbb{A}^+$: for any $\mathcal{U}, \mathcal{V} \in \mathbb{A}^+$, we write $\mathcal{U} \equiv_{m,r,k,l} \mathcal{V}$, if for any SO^{plog}-FO-sentence ϕ with $sqr(\phi) \leq m$, $\text{mva}(\phi) \leq r$, $\text{height}(\phi) \leq k$, and $fqr(\phi) \leq l$,

$$\mathcal{U} \models \phi \Longleftrightarrow \mathcal{V} \models \phi.$$

Lemma 3.3. For any natural numbers $m \geq 0$, $r, k, l \geq 1$, there exists $N \in \mathbb{N}$ such that for any strings $\mathcal{X}, \mathcal{Z} \in \mathbb{A}^*$ and $\mathcal{Y} \in \mathbb{A}^+$ and any natural number $h_1, h_2 > N$, if

$$\log(|\mathcal{X}| + h_1 \cdot |\mathcal{Y}| + |\mathcal{Z}|) = \log(|\mathcal{X}| + h_2 \cdot |\mathcal{Y}| + |\mathcal{Z}|),$$

then

$$\mathcal{X}\mathcal{Y}^{h_1}\mathcal{Z} \equiv_{m,r,k,l} \mathcal{X}\mathcal{Y}^{h_2}\mathcal{Z}.$$

The lemma is proved by a designed game for $\mathrm{SO}^{\mathrm{plog}}$-FO. It shows that $\mathrm{SO}^{\mathrm{plog}}$-FO cannot distinguish two strings with similar patterns and lengths. With this lemma, we can easily find some problems that separate MSO and $\mathrm{SO}^{\mathrm{plog}}$-FO. So we have:

Proposition 3.4. On strings, MSO $\not\leq$ $\mathrm{SO}^{\mathrm{plog}}$-FO and monadic $\Sigma_1^{\log}$-FO $\not\leq$ MSO.

Monadic $\Sigma_1^{\log}$-FO is a very weak fragment. Therefore $\mathrm{SO}^{\mathrm{plog}}$-FO $\not\leq$ MSO. Last but not least, by the "periodicity" of regular languages [3, 9], we also show that:

Theorem 3.5. If a language L is definable both in $\mathrm{SO}^{\mathrm{plog}}$-FO and MSO, the L is definable in FO.

References

[1] Liming Cai and Jianer Chen. On the amount of nondeterminism and the power of verifying. *SIAM Journal on Computing*, 26(3):733–750, 1997.

[2] Josep Diaz and Jacobo Torán. Classes of bounded nondeterminism. *Theory of Computing Systems*, 23(1):21–32, 1990.

[3] Heinz-Dieter Ebbinghaus and Jörg Flum. *Finite model theory*. Springer Science & Business Media, 2005.

[4] Ronald Fagin. Generalized first-order spectra and polynomial-time recognizable sets. *SIAM-AMS Proc.*, 7, 01 1974.

[5] Flavio Ferrarotti, SenÉn GonzÁles, Klaus-Dieter Schewe, and JosÉ MarÍa Turull-Torres. A restricted second-order logic for non-deterministic poly-logarithmic time. *Logic Journal of the IGPL*, 28(3):389–412, 2020.

[6] Erich Grädel, Phokion G Kolaitis, Leonid Libkin, Maarten Marx, Joel Spencer, Moshe Y Vardi, Yde Venema, Scott Weinstein, et al. *Finite Model Theory and its applications*. Springer, 2007.

[7] Neil Immerman. *Descriptive Complexity*. Springer Science & Business Media, 1998.

[8] Chandra Kintala and P Fischer. Refining nondeterminism in relativized complexity classes. *SIAM Journal on Computing*, 13(329-337):4, 1984.

[9] Howard Straubing. *Finite automata, formal logic, and circuit complexity*. Springer Science & Business Media, 2012.

[10] Kexu Wang and Xishun Zhao. A logic that captures βP on ordered structures. *Studies in Logic*, 13(3):01–18, 2020.