From quantifier depth to quantifier number: separating structures in finite variable logic

Harry Vinall-Smeeth
Humboldt-Universität zu Berlin
harry.vinall-smeeth@informatik.hu-berlin.de

Christoph Berkholz
Technische Universität Ilmenau
christoph.berkholz@tu-ilmenau.de

Overview Given two n-element structures, $\mathcal{A}$ and $\mathcal{B}$, which can be distinguished by a sentence of k-variable first-order logic, what is the minimum $f(n,k)$ such that there is guaranteed to be a sentence $\phi \in \mathcal{L}^k$ with at most $f(n,k)$ quantifiers, such that $\mathcal{A} \models \phi$ but $\mathcal{B} \not\models \phi$? We will present various results related to this question obtained by using the recently introduced QVT games [2], see *contributions* below. Through the lens of this question, we will highlight some difficulties that arise in analysing the QVT game and share some insights which can help to overcome them. A core theme is that of lifting quantifier depth lower bounds to quantifier number lower bounds.

Background The classic combinatorial game in finite model theory is the Ehrenfeucht–Fraïssé (EF) game. This game is played by two players, Spoiler and Duplicator, on a pair of structures $(\mathcal{A}, \mathcal{B})$. Spoiler tries to show the two structures are different, whilst Duplicator tries to show they are the same. The EF game captures the quantifier depth needed to separate $\mathcal{A}$ and $\mathcal{B}$ in the following sense: Spoiler can win the game in r-rounds if and only if there is a FO sentence ϕ with quantifier depth at most r, such that $\mathcal{A} \models \phi$ and $\mathcal{B} \not\models \phi$.

The quantifier depth needed to separate $\mathcal{A}$ and $\mathcal{B}$ therefore gives a measure of how different the two structures are. Recently a different measure, quantifier number, has received substantial attention [2–4]. One reason for this is that a combinatorial game has been discovered[1] which captures the number of quantifiers needed to separate two sets of structures: the Multi-Structural (MS) game. This game is similar to the EF game, with the following key differences. Firstly, the game is played on two sets of structures and secondly, Duplicator is given the power to make copies of structures. This has some novel consequences for how the game is played, see again [2–4].

Our work contributes to this line of research. In particular, we examine a trade-off between the number of variables and the number of quantifiers. This question has also been studied with the role of quantifier number replaced by quantifier depth [1,5]. In the aforementioned line of work a variant of EF games—which simultaneously captures the quantifier depth and the

[1] Actually the games were originally introduced by Immerman [6] but no further study of them was undertaken until they were rediscovered in [3].

number of variables needed to separate two structures—are crucial. An analogue to such games also exists in our setting: the quantifier-variable tree (QVT) game, recently introduced in [2].

Contributions To our knowledge there are no existing applications of the QVT game; our talk aims to highlight how such games can be analysed in practice. We will aim to draw attention to one key idea: that of lifting bounds on the quantifier depth to achieve bounds on the quantifier number. By proceeding in this way we can use results achieved in the context of quantifier depth—an area of research which has been far more deeply explored—as 'black boxes'.

Since we want to emphasise this aspect we will focus on the following two results, which are both obtained by lifting bounds on quantifier depth to bounds on quantifier number.

Result 1: An $O(n^{n^k})$ upper bound on the number of quantifiers needed to separate two n element structures by a sentence of $\mathcal{L}^k$.

Result 2: A $\Omega(2^{n^k})$ lower bound on the number of quantifiers needed to separate two n element structures by a *positive existential* sentence of $\mathcal{L}^k$.

We will focus on methodological aspects of our work that could be used in other contexts and may also be useful in improving our results. In this spirit, we will also introduce a game which we dub the k-lower bound game (k-LB game). This is simpler to work with than the QVT game but is still powerful enough to prove lower bounds on the number of quantifiers.

Talk Structure We will begin by introducing the QVT game. Interestingly this game is not played on two sets of structures but rather on a tree where each node is labelled with two sets of structures. We will discuss the reasons why this added layer of complication is necessary, via an example.

As a first application of the QVT games we will discuss Result 1. This result is easy and is a good opportunity to build some intuition. It also enables us to demonstrate how quantifier depth bounds may be transferred to quantifier number bounds in a simple way. The benefits of this approach can then be highlighted by showing how a recent quantifier depth upper bound [5] yields an improved upper bound on quantifier number.

Next, we will lay the groundwork for our presentation of Result 2 by showing how the QVT game can be adapted so that it captures the number of quantifiers in existential positive FO. We will then simplify things considerably by introducing our k-LB game. The cumulative effect is that we are left analysing a game played on two sets of structures where one of the sets of structures is a singleton.

It will then be time for the technical core of the talk: a discussion of Result 2. This lower bound is obtained via a construction which works as follows. Take two structures, $\mathcal{A}, \mathcal{B}$, which can be distinguished by a sentence of $\exists\mathcal{L}^k$. Then by adding elements and relations to these structures we can produce two new structures $S(\mathcal{A}), S(\mathcal{B})$ that can be distinguished by $\exists\mathcal{L}^{k+2}$. Moreover, if any sentence distinguishing $\mathcal{A}$ and $\mathcal{B}$ has quantifier depth r then any sentence distinguishing $S(A)$ and $S(B)$ has at least 2^r quantifiers. The exact details of the construction are quite involved but we will get the main ideas across via an extended example.

Finally, we will conclude by briefly discussing lower bounds that apply to FO, not just the positive existential fragment. We will discuss the difficulties of lifting Result 2 to this setting.

References

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