Indistinguishability games for Extensions of PDL with Intersection and Converse

DIEGO FIGUEIRA, Univ. Bordeaux, CNRS, Bordeaux INP, LaBRI, UMR 5800, F-33400 Talence, France
SANTIAGO FIGUEIRA, Univ. Buenos Aires, FCEN, DC & CONICET-UBA, ICC, Argentina
EDWIN PIN, Univ. Buenos Aires, FCEN, DM & CONICET-UBA, ICC, Argentina

We introduce a new kind of pebble games, which capture logical indistinguishability for programs and formulas of CPDL$^+$, a family of expressive logics rooted in Propositional Dynamic Logic (PDL). In terms of expressive power, CPDL$^+$ strictly contains PDL extended with intersection and converse (*a.k.a.* ICPDL) as well as Conjunctive Queries (CQ), Conjunctive Regular Path Queries (CRPQ), or some known extensions thereof (Regular Queries and CQPDL). The games introduced here are closely related to the "existential k-pebble game" [2, 3] on first-order structures with binary and unary signatures, but in our setting we add an additional rule stating that a pebble cannot be placed too far away from another pebble. This allows us to separate natural fragments of CPDL$^+$ that can be defined in terms of the tree-width of the underlying graphs of the expressions.

This extended abstract is based on the paper *PDL on Steroids: on Expressive Extensions of PDL with Intersection and Converse*, accepted at LICS'23.

AN EXTENSION OF CPDL

CPDL. The logic PDL was originally conceived for reasoning about programs [1]. However, variants of PDL are nowadays used in various areas of computer science, in particular in description logics, epistemic logics, program verification, or for querying datasets. CPDL expresses properties on (worlds and pairs of worlds of) *Kripke structures*, which are tuples of the form $K = (X, \{\rightarrow_a \mid a \in \mathbb{A}\}, \{X_p \mid p \in \mathbb{P}\})$ where X is a set of "*worlds*", $\rightarrow_a \subseteq X \times X$ is a transition relation for each $a \in \mathbb{A}$, and $X_p \subseteq X$ is a unary relation for each $p \in \mathbb{P}$. For K as above, we denote X by $W(K)$. *Expressions* of CPDL can be either *formulas* φ or *programs* π, defined by the following grammar, where p ranges over $\mathbb{P}$ and a over $\mathbb{A}$:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\pi\rangle$$
$$\pi ::= \varepsilon \mid a \mid \bar{a} \mid \pi \cup \pi \mid \pi \circ \pi \mid \pi^* \mid \varphi?$$

The semantics are given for programs $[\![\pi]\!]_K$ and for formulas $[\![\varphi]\!]_K$ in a Kripke structure K, where $[\![\pi]\!]_K \subseteq X \times X$ and $[\![\varphi]\!]_K \subseteq X$. The formal definition is not given here, but as usual, $[\![\pi]\!]_K$ represents the set of pairs of worlds of K where π holds, and $[\![\varphi]\!]_K$ represents the set worlds of K where φ holds. We write $K, u \models \varphi$ for $u \in [\![\varphi]\!]_K$ and $K, u, v \models \pi$ for $(u, v) \in [\![\pi]\!]_K$ and we write $\varphi_1 \equiv \varphi_2$ (resp. $\pi_1 \equiv \pi_2$) if $[\![\varphi_1]\!]_K = [\![\varphi_2]\!]_K$ (resp. $[\![\pi_1]\!]_K = [\![\pi_2]\!]_K$) for every structure K, in which case we say that φ_1, φ_2 (resp. π_1, π_2) are *equivalent*.

CPDL$^+$. An "*atom*" is an expression of the form $\pi(x, x')$, where π is a CPDL$^+$ program and $x, x' \in \mathbb{V}$. For an atom $\pi(x, x')$ we define $vars(\pi(x, x')) := \{x, x'\}$, and for a set of atoms C we define $vars(C) := \bigcup_{A \in C} vars(A)$. We define CPDL$^+$ as an extension of CPDL allowing also programs of the form

$$\pi ::= C[x_s, x_t]$$

where: (1) C is a finite set of atoms; (2) $x_s, x_t \in vars(C)$[1]; and (3) the *underlying graph* G_C of C is connected, where G_C has $vars(C)$ as vertices and $\{vars(A) \mid A \in C\}$ as edges. We call these

[1] Note that x_s and x_t may be equal or distinct variables.

programs *conjunctive programs*. We say that an expression of CPDL$^+$ is *positive* if it does not contain any subformula of the form $\neg\psi$. Observe that $\{x_s, x_t\} \subseteq vars(C)$, and hence we also define $vars(C[x_s, x_t]) := vars(C)$.

A function $f : vars(C) \to W(K)$, is a *C-satisfying assignment* if $(f(x), f(x')) \in [\![\pi']\!]_K$ for every atom $\pi'(x, x') \in C$. The semantics $[\![C[x_s, x_t]]\!]_K$ of a conjunctive program on a Kripke structure K is the set of all pairs $(w_s, w_t) \in W(K) \times W(K)$ such that $f(x_s) = w_s$ and $f(x_t) = w_t$ for some C-satisfying assignment f.

For any conjunctive program $\pi = C[x_s, x_t]$ we consider the *underlying graph* $G_{C[x_s,x_t]}$ of $C[x_s, x_t]$ as the graph having $vars(C)$ as vertices and $\{\{x_s, x_t\}\} \cup E(G_C)$ as edges. Observe that x_s and x_t are always connected via an edge in $G_{C[x_s,x_t]}$ but not necessarily in G_C. We define CPDL$^+$(TW$_k$) as the fragment of CPDL$^+$ whose only allowed conjunctive programs are of the form $C[x_s, x_t]$ where $G_{C[x_s,x_t]}$ has tree-width at most k.

INDISTINGUISHABILITY GAMES

In what follows we use the standard notation $\bar{u}$ to denote a tuple of elements from some set X and $\bar{u}[i]$ to denote the element in its i-th component. If $\bar{q} = (q_1, \dots, q_k)$ is a tuple and $1 \leq i \leq k$, by $\bar{q}[i \mapsto r]$ we denote the tuple $(q_1, \dots, q_{i-1}, r, q_{i+1}, \dots, q_k)$.

We will define the notion of k-simulation between pairs (K, K') of Kripke structures via a two-player zero-sum graph game $\mathbf{G}[\rightharpoonup_k]$. The arena of the game has a set of positions $S \cup D$, where $S = \{s\} \times Hom_k(K, K')$ and $D = \{d_1, \dots, d_k\} \times (W(K)^k \times W(K')^k)$ where Spoiler owns all positions from S and Duplicator all positions from D. The set of moves of $\mathbf{G}[\rightharpoonup_k]$ is the smallest set satisfying the following:[2]

(1) There is a move from $(s, \bar{u}, \bar{v})$ to $(d_i, \bar{u}', \bar{v})$ if $\bar{u}' = \bar{u}[i \mapsto w]$, where w is a world from K at distance ≤ 1 from $\bar{u}[j]$, for some $1 \leq j \leq k$ with $i \neq j$; and
(2) There is a move from $(d_i, \bar{u}', \bar{v})$ to $(s, \bar{u}', \bar{v}')$ if $\bar{v}' = \bar{v}[i \mapsto w]$, where w is a world from K' at distance ≤ 1 from $\bar{v}[j]$, for some $1 \leq j \leq k$ with $i \neq j$.

The winning condition for Duplicator is just any infinite play, which is a form of "Safety condition", which implies (positional) determinacy of the game. The 'pebbles' of each player are represented by $\bar{u}$ and $\bar{v}$ in the above definition and the rules of move say that each player can move a pebble provided that the following invariant is preserved: all pebbles are placed in the same connected component of each model.

Given two Kripke structures K, K', and tuples $\bar{v} \in W(K)^k$ and $\bar{v}' \in W(K')^k$, we say that $K', \bar{v}'$ *k-simulates* $K, \bar{v}$, notated $K, \bar{v} \rightharpoonup_k K', \bar{v}'$, if 1) $(s, \bar{v}, \bar{v}')$ is a valid position of $\mathbf{G}[\rightharpoonup_k]$ on (K, K') (*i.e.*, $\bar{v}, \bar{v}'$ induce a partial homomorphism), 2) all the worlds in $\bar{v}$ are in the same connected component of K, and 3) Duplicator has a winning strategy from $(s, \bar{v}, \bar{v}')$.

Theorem 1. *Let $k \geq 2$. Given Kripke structures K, K' where K' is of finite degree and worlds $u, v \in W(K)$ and $u', v' \in W(K')$, the following are equivalent*

(1) for every positive CPDL$^+$(TW$_k$)*-formula φ, we have $K, v \models \varphi$ implies $K', v' \models \varphi$; and*
(2) $K, v \rightharpoonup_{k+1} K', v'$;

and the following are equivalent

(1) for every positive CPDL$^+$(TW$_k$)*-program π, we have $K, u, v \models \pi$ implies $K', u', v' \models \pi$; and*
(2) $K, u, v \rightharpoonup_{k+1} K', u', v'$.

Furthermore, the hypothesis of finite degree is only needed for the 1*-to-*2 *implications.*

[2] For ease of notation we write $(s, \bar{u}, \bar{v})$ instead of $(s, (\bar{u}, \bar{v}))$ and the same for $(d_i, \bar{u}, \bar{v})$.

The notion of k-bisimulation on (K, K') is defined as before, using a two-player game $\mathbf{G}[\rightleftharpoons_k]$. The new rule (not formally given in this abstract) allows players to swap structures whenever Spoiler decides so, provided all pebbles in Spoiler's side are placed over the same world (and hence all pebbles in Duplicator's side are also placed over the same world). This accounts for the inclusion of negation of CPDL$^+$-formulas −observe that there is no negation of programs in CPDL$^+$.

We say that there is a *k-bisimulation* between $K, \bar{v}$ and $K', \bar{v}'$, and we note it $K, \bar{v} \rightleftharpoons_k K', \bar{v}'$ if 1) $(s, \bar{v}, \bar{v}')$ is a valid position of $\mathbf{G}[\rightleftharpoons_k]$ on (K, K') (*i.e.*, they induce partial homomorphisms), 2) all the worlds in $\bar{v}$ are in the same connected component of K and 3) Duplicator has a winning strategy from $(s, \bar{v}, \bar{v}')$ and also from $(s, \bar{v}', \bar{v})$.

Theorem 2. *Let $k \geq 2$. Given Kripke structures K, K' of finite degree and worlds $u, v \in W(K)$ and $u', v' \in W(K')$, the following are equivalent*

(1) for every CPDL$^+$(TW$_k$) *formula φ, we have $K, v \models \varphi$ iff $K', v' \models \varphi$; and*
(2) $K, v \rightleftharpoons_{k+1} K', v'$;

and the following are equivalent

(1) for every CPDL$^+$(TW$_k$) *program π, we have $K, u, v \models \pi$ iff $K', u', v' \models \pi$; and*
(2) $K, u, v \rightleftharpoons_{k+1} K', u', v'$.

Furthermore, the hypothesis of finite degree is only needed for the 1*-to-*2 *implications.*

There are two ways of defining a bisimulation $\rightleftharpoons$ for the full logic CPDL$^+$. One is simply to define it as the intersection of $\rightleftharpoons_k$ for all k, but in terms of a single game, we can also define $\rightleftharpoons$ analogously to $\rightleftharpoons_k$ but changing the rules of the game as follows: the number of pebbles is not fixed in advance; instead, Spoiler can add a new pebble in his turn, and Duplicator should answer with a similar move. Spoiler may choose to switch models provided all pebbles except one are removed, and so are the companion pebbles on the other side; in this case, the game restarts with one pebble in each structure with Spoiler and Duplicator having swapped structures.

APPLICATIONS

We showed that ICPDL, the extension of CPDL with 'intersection' (namely, adding $\pi ::= \pi \cap \pi$ with the obvious semantics) is equiexpressive to CPDL$^+$(TW$_1$) and also to CPDL$^+$(TW$_2$). Therefore we have defined a notion of (bi)similarity for ICPDL, which, to the best of our knowledge, was unknown.

As another application, we show the following result, which states a "tree-like model property":

Proposition 1. *For every $k \geq 2$, Kripke structure K, and world $u \in W(K)$, there exists a Kripke structure $\hat{K}$ of tree-width $\leq k-1$ and world $\hat{u} \in W(\hat{K})$ such that $K, u \rightleftharpoons_k \hat{K}, \hat{u}$. Further, if K is countable, $\hat{K}$ has a countable tree decomposition of width $\leq k-1$.*

As a consequence of the above result, we obtain

Theorem 3. *For every $k \geq 2$,* CPDL$^+$(TW$_{k+1}$) *is strictly more expressive than* CPDL$^+$(TW$_k$).

Essentially we show that the presence of a $(k+1)$-clique can be expressed in CPDL$^+$(TW$_k$) but not in CPDL$^+$(TW$_{k-1}$), for every $k \geq 3$.

REFERENCES

[1] Michael J. Fischer and Richard E. Ladner. 1979. Propositional Dynamic Logic of Regular Programs. *Journal of Computer and System Sciences (JCSS)* 18, 2 (1979), 194–211. https://doi.org/10.1016/0022-0000(79)90046-1
[2] Phokion G. Kolaitis and Moshe Y. Vardi. 1995. On the Expressive Power of Datalog: Tools and a Case Study. *Journal of Computer and System Sciences (JCSS)* 51, 1 (1995), 110–134. https://doi.org/10.1006/jcss.1995.1055
[3] Phokion G. Kolaitis and Moshe Y. Vardi. 2000. Conjunctive-Query Containment and Constraint Satisfaction. *Journal of Computer and System Sciences (JCSS)* 61, 2 (2000), 302–332. https://doi.org/10.1006/jcss.2000.1713