Quantifiers closed under partial polymorphisms

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1 Introduction

The k, n-bijection game was introduced by Hella [6] to characterize equivalence in the logic $L^k_{\infty\omega}(\mathbf{Q}_n)$, which is the extension of the infinitary logic with k variables by means of all n-ary Lindström quantifiers. Recall that a Lindström quantifier Q is specified by some isomorphism-closed class of relational structures over a fixed vocabulary σ. The quantifier is n-ary if all relations are of arity n or less. In particular, the $k, 1$-bijection game, often called the k-pebble bijection game, characterizes equivalence in $L^k_{\infty\omega}(\mathbf{Q}_1)$ which has the same expressive power as $C^k_{\infty\omega}$, the k-variable infinitary logic with counting. Hella uses the k, n-bijection game to show that, for each n, there is an $(n+1)$-ary quantifier that is not definable in $L^k_{\infty\omega}(\mathbf{Q}_n)$ for any k.

The $k, 1$-bijection game has been extensively used to establish inexpressibility results for $C^k_{\infty\omega}$. The k, n-bijection game for $n > 1$ has received relatively less attention. One reason is that while equivalence in $C^k_{\infty\omega}$ is a polynomial-time decidable relation, which is in fact a relation much studied on graphs in the form of the Weisfeiler-Leman algorithm, the relation induced by the k, n-bijection game for $n > 1$ reduces to isomorphism on graphs and is intractable in general. Nonetheless, there is some interest in studying, for example, the non-trivial equivalence induced by $L^k_{\infty\omega}(\mathbf{Q}_2)$ on structures with a ternary relation. Grochow and Levet [5] investigate this relation on finite groups.

A second reason why the logics $L^\omega_{\infty\omega}(\mathbf{Q}_n)$ have attracted less interest is that in finite model theory we are often interested in logics that are closed under first-order interpretations. This is especially so in Descriptive Complexity as the complexity classes we are trying to characterise usually have these closure properties. While $L^\omega_{\infty\omega}(\mathbf{Q}_1)$ is closed under first-order interpretations, this is not the case for $L^\omega_{\infty\omega}(\mathbf{Q}_n)$ for $n > 1$. Indeed, the closure of $L^\omega_{\infty\omega}(\mathbf{Q}_2)$ under interpretations already includes $\mathbf{Q}_n$ for all n and so can express all properties of finite structures.

One way of getting meaningful logics that include quantifiers of unbounded arity is to consider quantifiers which are invariant under stronger relations than isomorphism. As an example, the class of linear-algebraic quantifiers, introduced in [3], is characterized by the invertible map games introduced in [4]. These games are used in a highly sophisticated way by Lichter [8] to demonstrate a polynomial-time property that is not definable in fixed-point logic with rank. The result is extended to the infinitary logic with all linear-algebraic quantifiers in [2].

Another example is the recent result of Hella [7] showing a hierarchy result for CSP quantifiers, using a novel game. Recall that for a fixed relational structure $\mathfrak{B}$, $\mathsf{CSP}(\mathfrak{B})$ denotes the class of structures that map homomorphically to $\mathfrak{B}$. Hella establishes that, for each $n > 1$, there is a structure $\mathfrak{B}$ with $n+1$ elements that is not definable in $L^\omega_{\infty\omega}(\mathbf{Q}_1, \mathbf{CSP}_n)$, where $\mathbf{CSP}_n$ denote the collection of all quantifiers of the form $\mathsf{CSP}(\mathfrak{B}')$ where $\mathfrak{B}'$ has at most n elements. Note that $\mathbf{CSP}_n$ includes quantifiers of all arities.

The interest in CSP quantifiers is inspired by the great progress that has been made in classifying constraint satisfaction problems in recent years, resulting in the dichotomy theorem of Bulatov and Zhuk [1, 9]. The so-called algebraic approach to the classification of CSP has shown that the complexity of $\mathsf{CSP}(\mathfrak{B})$ is completely determined by the polymorphisms of the structure $\mathfrak{B}$, which in turn determine certain closure properties for the class of structures $\mathsf{CSP}(\mathfrak{B})$.

Motivated by this, we introduce certain closure properties of classes of structures, which we call *partial polymorphisms*. These closure properties generalize the closure conditions obtained on $\mathsf{CSP}(\mathfrak{B})$ from the polymorphisms of $\mathfrak{B}$. We show how these closure conditions naturally give rise to a two-player game. More specifically, for each invariant family $\mathcal{P}$ of partial polymorphisms, there is a k-pebble game played on pairs of structures $(\mathfrak{A}, \mathfrak{B})$ such that a Duplicator winning strategy in this game guarantees that $\mathfrak{A}$ and $\mathfrak{B}$ cannot be distinguished in $L^k_{\infty\omega}(\mathbf{Q}_\mathcal{P})$ where $\mathbf{Q}_\mathcal{P}$ is the collection of all quantifiers that are closed under the family $\mathcal{P}$.

We use this game, and a construction in the style of Cai, Fürer and Immerman, to give a class of structures that is not definable in $L^k_{\infty\omega}(\mathbf{Q}_{\mathcal{N}_\ell})$ where $\mathcal{N}_\ell$ is the collection of *near-unanimity* partial polymorhisms of arity ℓ. We illustrate how the class of quantifiers $\mathbf{Q}_{\mathcal{N}_\ell}$ is considerably more general than the class of CSP which have a near-unanimity polymorphism

In the rest of this abstract, we give the fundamental definitions of closure under partial polymorphisms and of the corresponding pebble game.

2 Partial Polymorphisms

Let t and t' be constant terms (possibly) containing partial functions. Then the identity $t \simeq t'$ means that either both t and t' are undefined, or they are both defined and their value is the same.

Definition 1 *Let $A \neq \emptyset$ be a set, and let p be a be a partial function $A^n \to A$.*

(a) If $\vec{a}_i = (a_i^1, \ldots, a_i^r) \in A^r$ for each $i \in [n]$ and $p(\vec{a}^j)$ is defined for each $j \in [r]$, where $\vec{a}^j := (a_1^j, \ldots, a_n^j)$, then applying p column wise, we obtain the new tuple $\hat{p}(\vec{a}_1, \ldots, \vec{a}_n) := (p(\vec{a}^1), \ldots, p(\vec{a}^r))$.

(b) If $R \subseteq A^r$, then applying p column wise to it we obtain the relation $\hat{p}(R) := \{\hat{p}(\vec{a}_1, \ldots, \vec{a}_n) \mid \vec{a}_1, \ldots, \vec{a}_n \in R\}$.

(c) If $\mathfrak{A} = (A, R_1^\mathfrak{A}, \ldots, R_m^\mathfrak{A})$ is a structure, we denote the structure $(A, \hat{p}(R_1^\mathfrak{A}), \ldots, \hat{p}(R_m^\mathfrak{A}))$ by $\hat{p}(\mathfrak{A})$.

Definition 2 *Suppose p_A is a partial function $A^n \to A$ for each finite (nonempty) set $A \subseteq \omega$. We say that the family $\{p_A \mid A \in \mathcal{P}_{\text{fin}}(\omega)\}$ is* invariant *if it respects bijections: if $f \colon A \to B$ is a bijection and $a_1, \ldots, a_n \in A$, then $p_B(f(a_1), \ldots, f(a_n)) \simeq f(p_A(a_1, \ldots, a_n))$.*

Note that if the family $\{p_A \mid A \in \mathcal{P}_{\text{fin}}(\omega)\}$ is invariant and $|A| > |\{a_1, \ldots, a_n\}| + 1$, then $p_A(a_1, \ldots, a_n)$ is either undefined, or a_i for some $i \in [n]$. Moreover, the subscript i (or being undefined) is completely determined by the equality type of the tuple $(a_1, \ldots, a_n)$. Another way of stating this is that the family is invariant if it can be defined (as a relation) in the language of equality.

Definition 3 *For τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, we write $\mathfrak{A} \leq \mathfrak{B}$ if $A = B$ and $R^\mathfrak{A} \subseteq R^\mathfrak{B}$ for each $R \in \tau$.*

Definition 4 *Let $\mathcal{P}$ be an invariant family of n-ary partial functions, and let $Q_\mathcal{K}$ be a generalized quantifier. We say that $Q_\mathcal{K}$ is $\mathcal{P}$-closed if the following holds for all $\mathfrak{A}$ and $\mathfrak{B}$:*

if $\mathfrak{B} \in \mathcal{K}$ and $\mathfrak{A} \leq \hat{p}_A(\mathfrak{B})$, then $\mathfrak{A} \in \mathcal{K}$.

We denote the class of $\mathcal{P}$-closed quantifiers by $\mathbf{Q}_\mathcal{P}$.

Example 5 *Let $\mathcal{N}_\ell$ be the family of partial functions $n_A \colon A^\ell \to A$ s.t. $n_A(\vec{a}) = a$ whenever $a_i \neq a$ for at most one $i \in [\ell]$, and $n_A(\vec{a})$ is undefined otherwise. The hypergraph coloring quantifiers $Q_{\mathsf{CSP}(H_{n,m})}$ are $\mathcal{N}_3$-closed, where $H_{n,m}$ is the complete hypergraph of size n and edges of size m.*

3 Pebble Games

Let $\mathcal{P}$ be an invariant family of partial polymorphisms. Given two structures $\mathfrak{A}$ and $\mathfrak{B}$ of the same vocabulary, and assignments α and β on $\mathfrak{A}$ and $\mathfrak{B}$, respectively, such that $\text{dom}(\alpha) = \text{dom}(\beta) \subseteq X_k$, where $X_k = \{x_1, \dots, x_k\}$. we write $(\mathfrak{A}, \alpha) \equiv^k_{\infty\omega,\mathcal{P}} (\mathfrak{B}, \beta)$ if $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ satisfy the same $L^k_{\infty\omega}(\mathbf{Q}_{\mathcal{P}})$-formulas. If $\alpha = \beta = \emptyset$, we write simply $\mathfrak{A} \equiv^k_{\infty\omega,\mathcal{P}} \mathfrak{B}$ instead of $(\mathfrak{A}, \emptyset) \equiv^k_{\infty\omega,\mathcal{P}} (\mathfrak{B}, \emptyset)$.

The *k-pebble* $\mathcal{P}$ *game* for $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ is played between *Spoiler* and *Duplicator*. We denote the game by $\mathrm{PG}^{\mathcal{P}}_k(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$, and we use the shorthand notation $\mathrm{PG}^{\mathcal{P}}_k(\alpha, \beta)$ whenever $\mathfrak{A}$ and $\mathfrak{B}$ are clear from the context.

Definition 6 *The rules of the game* $\mathrm{PG}^{\mathcal{P}}_k(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$ *are the following:*

1. *If* $\alpha \mapsto \beta$ *is not a partial isomorphism, then the game ends, and Spoiler wins.*
2. *If (1) does not hold, there are two types of moves that Spoiler can choose to play:*
   - **Left $\mathcal{P}$-quantifier move:** *Spoiler starts by choosing* $r \in [k]$ *and an r-tuple* $\vec{y} \in X^r_k$ *of distinct variables. Duplicator chooses next a bijection* $f \colon A \to B$*. Spoiler answers by choosing an r-tuple* $\vec{b} \in B^r$*. Duplicator answers by choosing r-tuples* $\vec{a}_1, \dots, \vec{a}_n$ *such that* $f^{-1}(\vec{b}) = \hat{p}_A(\vec{a}_1, \dots, \vec{a}_n)$*. Spoiler completes the round by choosing* $i \in [n]$*. The players continue by playing* $\mathrm{PG}^{\mathcal{P}}_k(\alpha[\vec{a}_i/\vec{y}], \beta[\vec{b}/\vec{y}])$*.*
   - **Right $\mathcal{P}$-quantifier move:** *Similar to the above except that Spoiler chooses a tuple in* A^r *and Duplicator responds by a sequence of r-tuples from* B*.*
3. *Duplicator wins the game if Spoiler does not win it in a finite number of rounds.*

Theorem 7 *If Duplicator has a winning strategy in* $\mathrm{PG}^{\mathcal{P}}_k(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$*, then* $(\mathfrak{A}, \alpha) \equiv^k_{\infty\omega,\mathcal{P}} (\mathfrak{B}, \beta)$*.*

Using a version of the Cai-Fürer-Immerman construction we prove that for all ℓ and k there are non-isomorphic structures $\mathfrak{A}$ and $\mathfrak{B}$ such that Duplicator has a winning strategy in $\mathrm{PG}^{\mathcal{N}_\ell}_k(\mathfrak{A}, \mathfrak{B})$. On the other hand, $\mathfrak{A}$ and $\mathfrak{B}$ can be separated by a polynomial time CSP. This shows that $\equiv^k_{\infty\omega,\mathcal{N}_\ell}$ does not capture isomorphism for any k, and $L^\infty_{\infty\omega}(\mathbf{Q}_{\mathcal{N}_\ell})$ does not capture all of PTIME.

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